Solving a Fuzzy Initial Value Problem of a Harmonic Oscillator Model

by M. Ahsar Karim

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Abstract. Modeling in systems biology is often faced with challenges in terms of measurement uncertainty. This is possibly either due to limitations of available data, environmental or demographic changes. One of typical behavior that commonly appears in the systems biology is a periodic behavior. Since uncertainties would get involved into the systems, the change of solution behavior of the periodic system should be taken into account. To get insight into this issue, in this work a simple mathematical model describing periodic behavior, i.e. a harmonic oscillator model, is considered by assuming its initial value has uncertainty in terms of fuzzy number. The system is known as Fuzzy Initial Value Problems. Some methods to determine the solutions are discussed. First, solutions are examined using two types of fuzzy differentials, namely Hukuhara Differential (HD) and Generalized Hukuhara Differential (GHD). Application of fuzzy arithmetic leads that each type of HD and GHD are formed into α-cut deterministic systems, and then are solved by the Runge-Kutta method. The HD type produces a solution with increasing uncertainty starting from the initial condition. While, GHD type produces an oscillatory solution but only until a certain time and above it the uncertainty becomes monotonic increasing. Solutions of both types certainly do not provide the accuracy for harmonic oscillator model during its evolution. Therefore, we propose the third method, so called Fuzzy Differential Inclusions (FDI), to attack the problem. Using this method, we obtain oscillatory solutions during its evolution.

INTRODUCTION

One of very important aspect in the study of systems biology is the concept of modeling the dynamics of biochemical networks. The large size and complexity [1] of these networks are often become to major problems to fully understand their dynamic behavior. One of the typical behavior that is often faced is a periodic behavior. In addition to this, the limitations of available technology and the changes in environmental conditions or demographic factors could have impacts on the measurement uncertainty. Therefore, a mathematical description that can accommodate the uncertainty and the oscillatory behavior issues is needed in order to do analysis and predictions on the behavior of the system. To get insight into these issues a, a Fuzzy Initial Value Problem (FIVP) of a Harmonic Oscillator Model (HOM) is considered,

$$\tilde{y}_1' = \tilde{y}_2,\ \tilde{y}_2' = -k_1\tilde{y}_1 - k_2\tilde{y}_2,\ \tilde{y}_1(0) = \tilde{y}_{1\_0},\ \tilde{y}_2(0) = \tilde{y}_{2\_0} \qquad (1)$$

with $\tilde{y}_{1\_0}, \tilde{y}_{2\_0} \in \Im_{\wp}(R)$ are initial conditions with $\mathrm{supp}(\tilde{y}_{1\_0}), \mathrm{supp}(\tilde{y}_{2\_0}) \subseteq R^+$ and k_1, k_2 are the real constant parameters. Here, $\Im_{\wp}(R)$ is the family of all the fuzzy numbers on R. Two important things will then be studied, namely application of the concept of fuzzy differentials and the solution methodology of the FIVP.

Some of the concepts of differential equations that accommodate an uncertainty has been introduced by [2, 3, and 4], so called Fuzzy Differential Equations (FDE). The first proposal was given by Hukuhara, which is called by Hukuhara differential (HD) [2, 3, 4]. The HD concept was derived from the concept of interval-valued functions. Moreover, Seikkala proposed the concept based on the concept of α-cut, called as Seikkala derivative [2, 3, 4], which was later proved to be equivalent to HD concept. Both concepts were then expanded into a so called

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Generalized Hukuhara Differentials (GHD) [2, 3, 4]. The other concept was introduced by Baidosov, known as the concept of fuzzy differential inclusions (FDI) [5, 6, 7]. The HD, GHD, and FDI concepts transform the fuzzy problem into α-cut deterministic models. The solutions of the α-cut deterministic models, hereinafter called fuzzy solutions, are determined by using Runge-Kutta method.

FUZZY CONCEPTS

Fuzzy Arithmetic

To be self-contained, some concepts of fuzzy arithmetic are here introduced. Here, X is a classical non-empty set and R is a set of all real numbers.

Definition 1. [8, 9, 10]. A fuzzy subset F of X is described by a function $F: X \to [0,1]$, called membership function of X. The value $F(x) \in [0,1]; \forall x \in X$ indicates the membership degree of the element x of X in fuzzy set F.

Definition 2. [8, 9, 10] The α-cut of a fuzzy subset F, denoted by $[F]^\alpha$, is the set of all elements that belong to a fuzzy set F with at least α degree, that is, $[F]^\alpha = \{x \in X : F(x) \geq \alpha\}, \alpha \in [0,1]$.

Definition 3. [8, 9, 10] A fuzzy set F is called by fuzzy number, if

(a) F is normal, that is, $\exists x \in R \ni F(x) = 1$.

(b) $[F]^\alpha, \forall \alpha \in [0,1]$ are closed intervals.

(c) Support of F, that is $\text{Supp}(F) = \{x \in R : F(x) > 0\}$ are bounded.

The collection of all fuzzy numbers F of R denoted by R_F, and the α-cut of the fuzzy numbers F shortened by $[F]^\alpha = [F_\alpha^-, F_\alpha^+]$, with $F_\alpha^- = \inf\{x \in R : F(x) \geq \alpha\}$ and $F_\alpha^+ = \sup\{x \in R : F(x) \geq \alpha\}$.

Definition 4. [8, 9, 10] Let A and B be fuzzy numbers with α-cuts $[A]^\alpha = [A_\alpha^-, A_\alpha^+]$ and $[B]^\alpha = [B_\alpha^-, B_\alpha^+]$, respectively, and a real number δ.

(a) The sum and the difference of $[A]^\alpha$ and $[B]^\alpha$:

$[A+B]^\alpha = [A]^\alpha + [B]^\alpha = [A_\alpha^- + B_\alpha^-, A_\alpha^+ + B_\alpha^+]$ and $[A-B]^\alpha = [A]^\alpha - [B]^\alpha = [A_\alpha^- - B_\alpha^+, A_\alpha^+ - B_\alpha^-]$.

(b) The multiplication of $[A]^\alpha$ by δ:

$$[\delta A]^\alpha = \delta[A]^\alpha = \delta[A_\alpha^-, A_\alpha^+] = \begin{cases} [\delta A_\alpha^-, \delta A_\alpha^+]; \delta \geq 0 \\ [\delta A_\alpha^+, \delta A_\alpha^-]; \delta < 0 \end{cases}$$

(c) The multiplications of $[A]^\alpha$ and $[B]^\alpha$:

$[A \cdot B]^\alpha = [A]^\alpha \cdot [B]^\alpha = [\min P, \max P]; P = \{A_\alpha^- B_\alpha^-, A_\alpha^- B_\alpha^+, A_\alpha^+ B_\alpha^-, A_\alpha^+ B_\alpha^+\}$

(d) The division of $[A]^\alpha$ by $[B]^\alpha$, if $0 \notin \text{supp}(B)$:

$[A/B]^\alpha = [A]^\alpha / [B]^\alpha = [A_\alpha^-, A_\alpha^+] \cdot [1/B_\alpha^+, 1/B_\alpha^-]$

Fuzzy Differential Equations (FDE)

Some basic concepts of FDE, i.e. HD and GHD, will be presented below.

Definition 5. [2, 3, 4] Let $F: (a,b) \to R_F$; $(a,b) \subseteq R$ be a fuzzy function. Then $[F'(x)]^\alpha = [(F'(x))_\alpha^-, (F'(x))_\alpha^+]$; $\forall \alpha \in [0,1]$ and $F'(x) \in R_F$, is called by Seikkala derivative of F. The fuzzy function F is called by Seikkala differentiable.

Lemma 1. [2, 3, 4] Let $F, G: (a,b) \to R_F$; $(a,b) \subseteq R$ be a fuzzy functions. If F and G are Seikkala differentiable, respectively, then $(F+G)' = F' + G'$ and $(kF)' = k F'$, $\forall k \in R$.

Definition 6. [3] Let $\Im_\wp(R)$ be the family of all the fuzzy numbers on R and the function $F: (a,b) \to \Im_\wp(R)$. If the limits of some pair

(a) $\lim_{h\to 0^+} \frac{F(x_0+h)-_H F(x_0)}{h}$ and $\lim_{h\to 0^+} \frac{F(x_0)-_H F(x_0-h)}{h}$ or

(b) $\lim_{h\to 0^+} \frac{F(x_0)-_H F(x_0+h)}{-h}$ and $\lim_{h\to 0^+} \frac{F(x_0-h)-_H F(x_0)}{-h}$

exists and are equal to some element $F'(x_0)\in \Im_\wp(R)$, then F is strongly generalized differentiable at x_0 and $F'(x_0)$ is the strongly generalized derivative of F at x_0.

The Hukuhara difference $-_H$ has rules: if $A, B \in \Im_\wp(R)$, then $A -_H B = C \Leftrightarrow A = B + C$, if C exist with $+$ is the standard addition operation on fuzzy numbers (see Definition 4). In the term of α-cut, $[A-_H B]^\alpha = [a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+]$ with $[A]^\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]^\alpha = [b_\alpha^-, b_\alpha^+]$.

If the function F satisfy the Definition 6 (a) then F is called by Hukuhara Differentiable (HD), and if the F satisfy the Definition 6 (b) then F is called by Generalized Hukuhara Differentiable (GHD).

Lemma 2. [3] Let $F:(a,b)\to \Im_\wp(R)$. If $F(x) = (f_\alpha^-(x), f_\alpha^+(x)) \in \Im_\wp(R)$, then

(a) If F is HD, then $F' = (f_\alpha^{-}{}', f_\alpha^{+}{}')$

(b) If F is GHD, then $F' = (f_\alpha^{+}{}', f_\alpha^{-}{}')$.

Fuzzy Differential Inclusions (FDI)

As generalization of a differential inclusion, FDI is defined as [5, 6]

$$y'(t) \in F(t, y(t)), y(0) \in \tilde{y}_0 \tag{2}$$

and also considered as the family of differential inclusions:

$$y'(t) \in [F(t, y(t))]^\alpha, y(0) \in [\tilde{y}_0]^\alpha \tag{3}$$

for all $\alpha \in [0,1]$, where $[F]^\alpha : [0,T] \times R^n \to \Im_\wp(R)$ and $[\tilde{y}_0]^\alpha \in \Im_\wp(R)$. The solution of the Problem (3) is a continuous function $y:[0,T]\to R^n$ that satisfies the inclusion, i.e. in $[0,T]$ and $y(0) = y_0 \in [\tilde{y}_0]^\alpha$. Diamond [7] has proved that the set of all solutions of Problem (3) are the α-cuts of the fuzzy solution of Problem (2). On the other hand, Gomes [3] has proved that, if F is continuous and bounded, then all the solutions to Problem (3) are defined.

RESULTS AND DISCUSSION

Consider Equation (1). Since the initial condition $\tilde{y}_{1\_0}, \tilde{y}_{2\_0}$ are fuzzy numbers, then the solutions must be fuzzy valued continuous functions $\tilde{y}_1(t), \tilde{y}_2(t)$ at any given t. Let the α-cut of $\tilde{y}_1$ and $\tilde{y}_2$:

$$[\tilde{y}_1]^\alpha = [y_{1\alpha}^-, y_{1\alpha}^+] \quad \text{and} \quad [\tilde{y}_2]^\alpha = [y_{2\alpha}^-, y_{2\alpha}^+]. \tag{4}$$

In order to provide an illustration, let the initial conditions as *Triangular Membership Function* ([8],[10]), one of the forms of fuzzy numbers, i.e. $\tilde{y}_{1\_0} = A = \text{trimf}(x,[0,1,2])$ and $\tilde{y}_{2\_0} = B = \text{trimf}(x,[1,2,3])$, with the α-cuts:

$$[\tilde{y}_{1\_0}]^\alpha = [y_{1\_0\alpha}^-, y_{1\_0\alpha}^+] = [0.5, 1.5] \quad \text{and} \quad [\tilde{y}_{2\_0}]^\alpha = [y_{2\_0\alpha}^-, y_{2\_0\alpha}^+] = [1.5, 2.5], \tag{5}$$

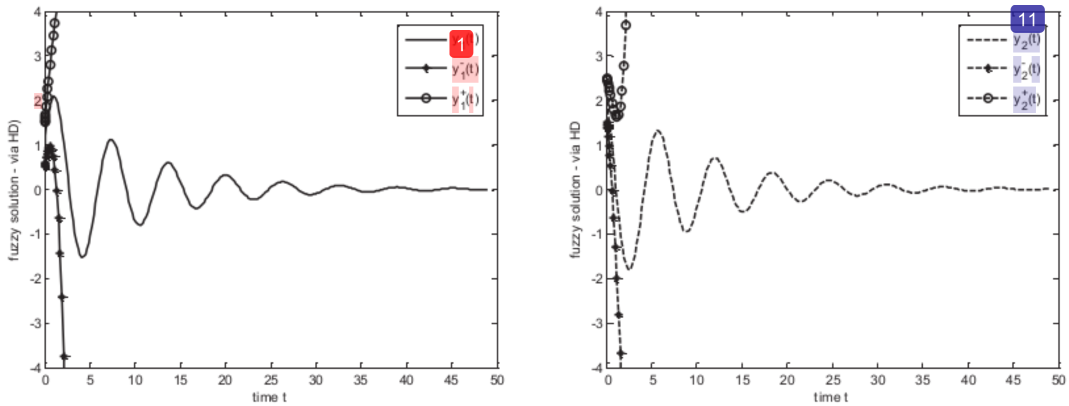
for $\alpha = 0.5$. Two behaviors will be addressed: decaying and harmonic Free Vibrations.

Model and Solution using HD concept

Using the HD concept, the α-cut deterministic systems ares given by

$$\begin{array}{llll} y_{1\alpha}^-{}' = y_{2\alpha}^- & y_{2\alpha}^-{}' = -k_1 y_{1\alpha}^+ - k_2 y_{2\alpha}^+ & y_{1\_0\alpha}^- = 0.5 & y_{2\_0\alpha}^- = 1.5 \\ y_{1\alpha}^+{}' = y_{2\alpha}^+ & y_{2\alpha}^+{}' = -k_1 y_{1\alpha}^- - k_2 y_{2\alpha}^- & y_{1\_0\alpha}^+ = 1.5 & y_{2\_0\alpha}^+ = 2.5 \end{array} \tag{6}$$

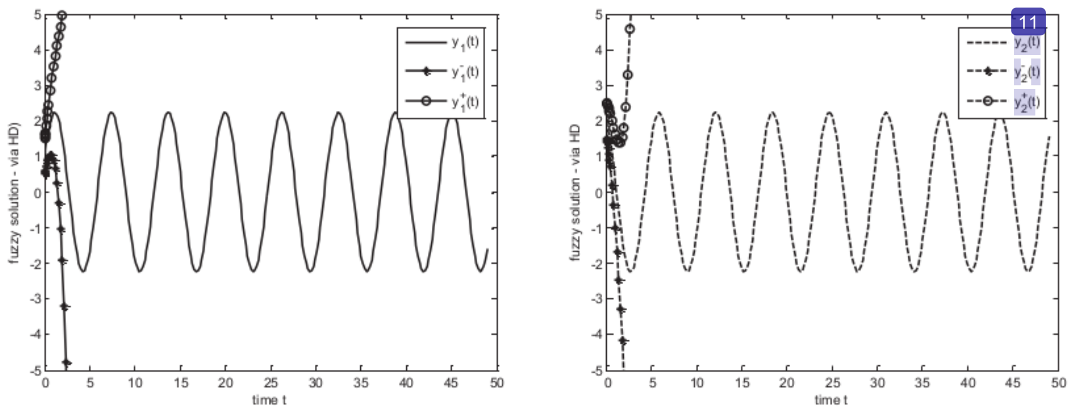
Taking $k_1 = 1$, $k_2 = 0.2$ and then solving (6) using the Runge-Kutta method, solutions are shown in Fig 1.



(1.a) (1.b)

FIGURE 1. The α-cut of fuzzy solutions of the Problem (6) using HD concept, with the parameters $k_1 = 1$, $k_2 = 0.2$. (1.a) $y_{1\alpha}^-(t)$ is denoted by the asterisk and $y_{1\alpha}^+(t)$ is by the circle mark; a crips solution $y_1(t)$ by the solid line. (1.b) $y_{2\alpha}^-(t)$ is denoted by the asterisk and $y_{2\alpha}^+(t)$ by the circle mark; a crips solution $y_2(t)$ by the dashed line.

Next, taking $k_1 = 1$, $k_2 = 0$ and then using the Runge-Kutta method, results are shown in Fig 2.



(2.a) (2.b)

FIGURE 2. The α-cut of fuzzy solutions of the Problem (6) using HD method, with the parameters $k_1 = 1$, $k_2 = 0$. Information for (2.a) and (2.b) are equivalent to (1.a) and (1.b) in Fig 1, respectively.

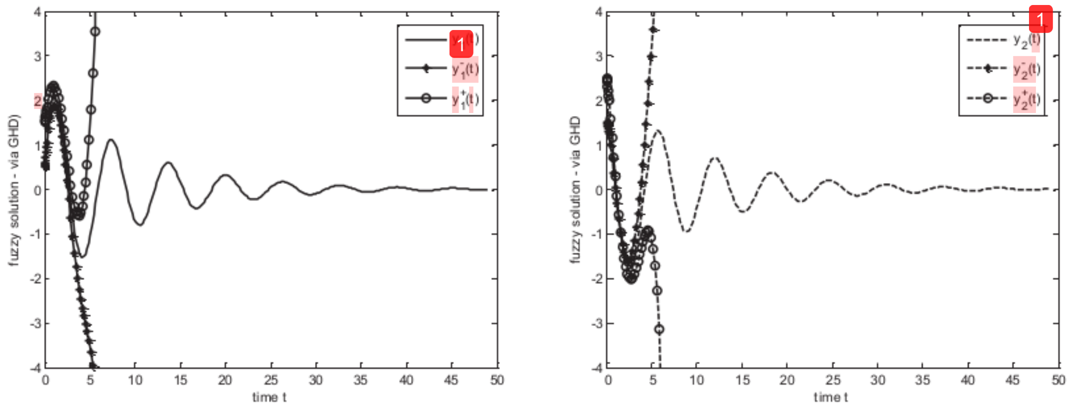
The fuzzy solutions $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$ which were obtained by the concept of HD (Fig 1 and Fig 2) did not show oscillations as shown by crips solutions (solutions without uncertainty effect); they increase very quickly. This means that the HD concept cannot capture oscillatroy behavior.

Model and Solution using GHD concept

Using the GHD concept, equations are now

$$\begin{matrix} y_{1\alpha}^-{}' = y_{2\alpha}^+ & y_{2\alpha}^-{}' = -k_1 y_{1\alpha}^- - k_2 y_{2\alpha}^- & y_{1\_0\alpha}^- = 0.5 & y_{2\_0\alpha}^- = 1.5 \\ y_{1\alpha}^+{}' = y_{2\alpha}^- & y_{2\alpha}^+{}' = -k_1 y_{1\alpha}^+ - k_2 y_{2\alpha}^+ & y_{1\_0\alpha}^+ = 1.5 & y_{2\_0\alpha}^+ = 2.5 \end{matrix} \quad (7)$$

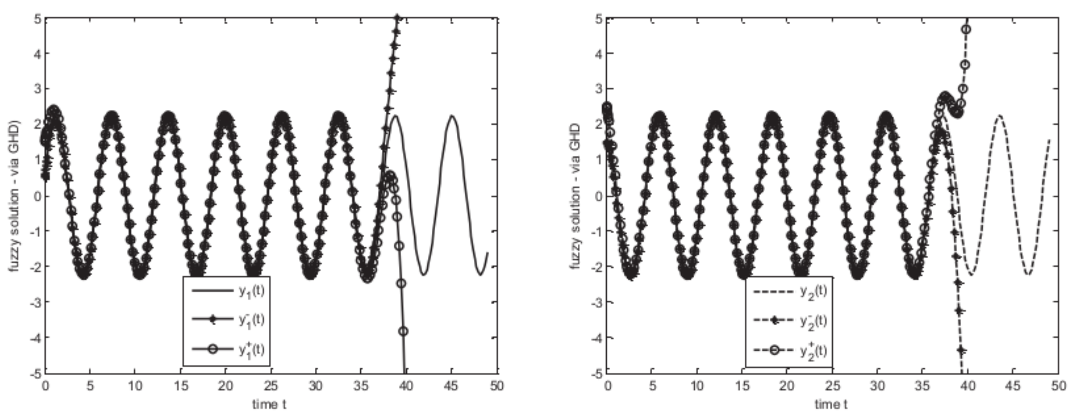
Using similar parameter values as in Section 3.1, results are given by Fig 3-4.



(3.a) (3.b)

FIGURE 3. Same information as in Fig 1 but now for GHD concept.

Results in Fig 3 show that the fuzzy solutions $\tilde{y}_1(t)$ has non-decreasing diameter while $\tilde{y}_2(t)$ has non-increasing diameter. This leads to the existence of switch point in which the lower bound α-cut solution will then take over to be the upper bound α-cut solution; this would not be possible in reality.



(4.a) (4.b)

FIGURE 4. Same information as in Fig 2, but now for GHD concept.

Results in Fig 4 show that application of GHD concept can maintain the diameters of α-cuts of the fuzzy solutions to some periods, but then the fuzzy solutions increase rapidly. To conclude, GHD concept cannot capture the whole oscillatroy behavior.

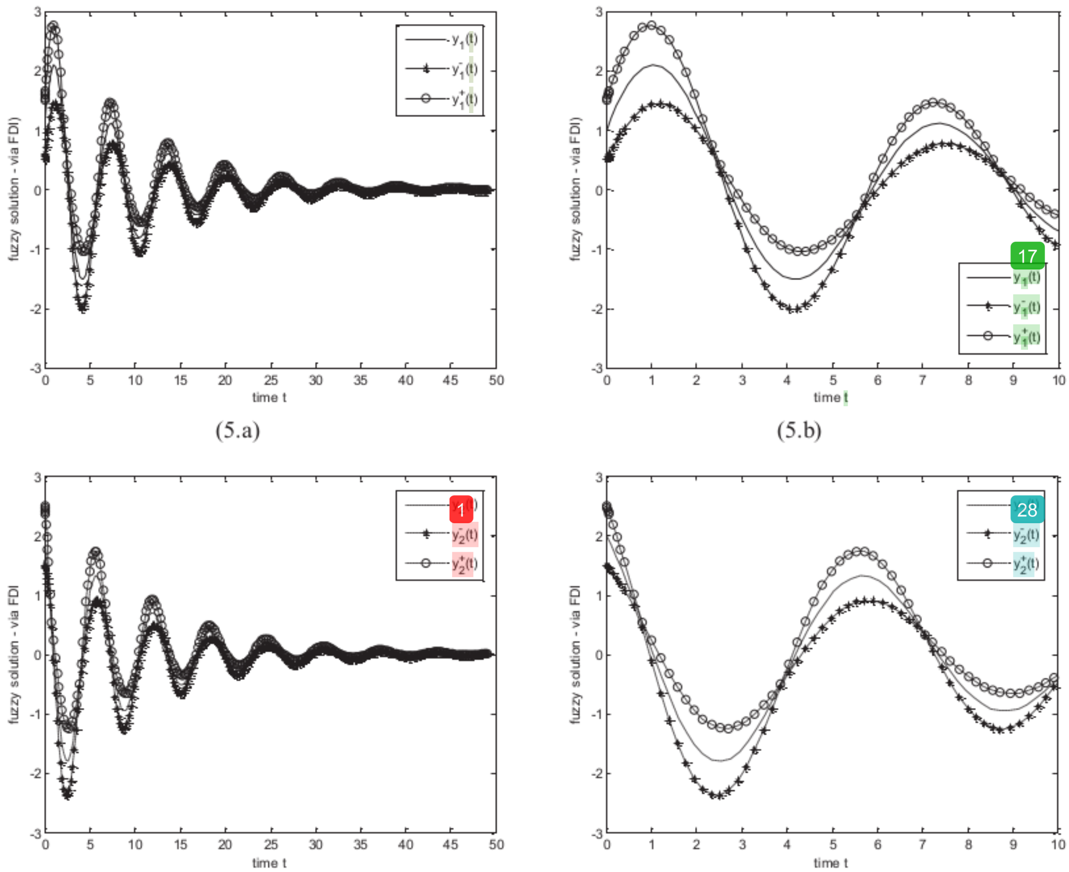
Model and Solution using FDI concept

Here, the α-cut solutions are obtained as

$$[\tilde{y}(t)]^{\alpha} = ([\tilde{y}_1(t)]^{\alpha}, [\tilde{y}_2(t)]^{\alpha}) = ([\min y_1(t), \max y_1(t)], [\min y_2(t), \max y_2(t)]), \quad (8)$$

with $y_{1\_0} \in \{0.5, 1.5\}$ and $y_{2\_0} \in \{1.5, 2.5\}$. Results are shown in Fig 5-6.

Using this concept, the oscillatroy behavior can be captured. The existence of the functions "min" and "max" guarantee the continuity and the boundedness of functions $y_1(t) \in [y_{1\alpha}^-(t), y_{1\alpha}^+(t)]$ and $y_2(t) \in [y_{2\alpha}^-(t), y_{2\alpha}^+(t)]$.



(5.a) (5.b)

(5.c) (5.d)

FIGURE 5. The α-cut of fuzzy solutions of the Problem (8) using FDI concept, with the parameters $k_1 = 1$, $k_2 = 0.2$. (5.a) $y_{1\alpha}^-(t)$ is denoted by the asterisk and $y_{1\alpha}^+(t)$ is by the circle mark; a crips solution $y_1(t)$ by the solid line. (5.b) As in (5.a) but now for $t \in [0,10]$. (5.c) $y_{2\alpha}^-(t)$ is denoted by the asterisk and $y_{2\alpha}^+(t)$ by the circle mark; a crips solution $y_2(t)$ by the dashed line. (5.d) As in (5.c) but now for $t \in [0,10]$.

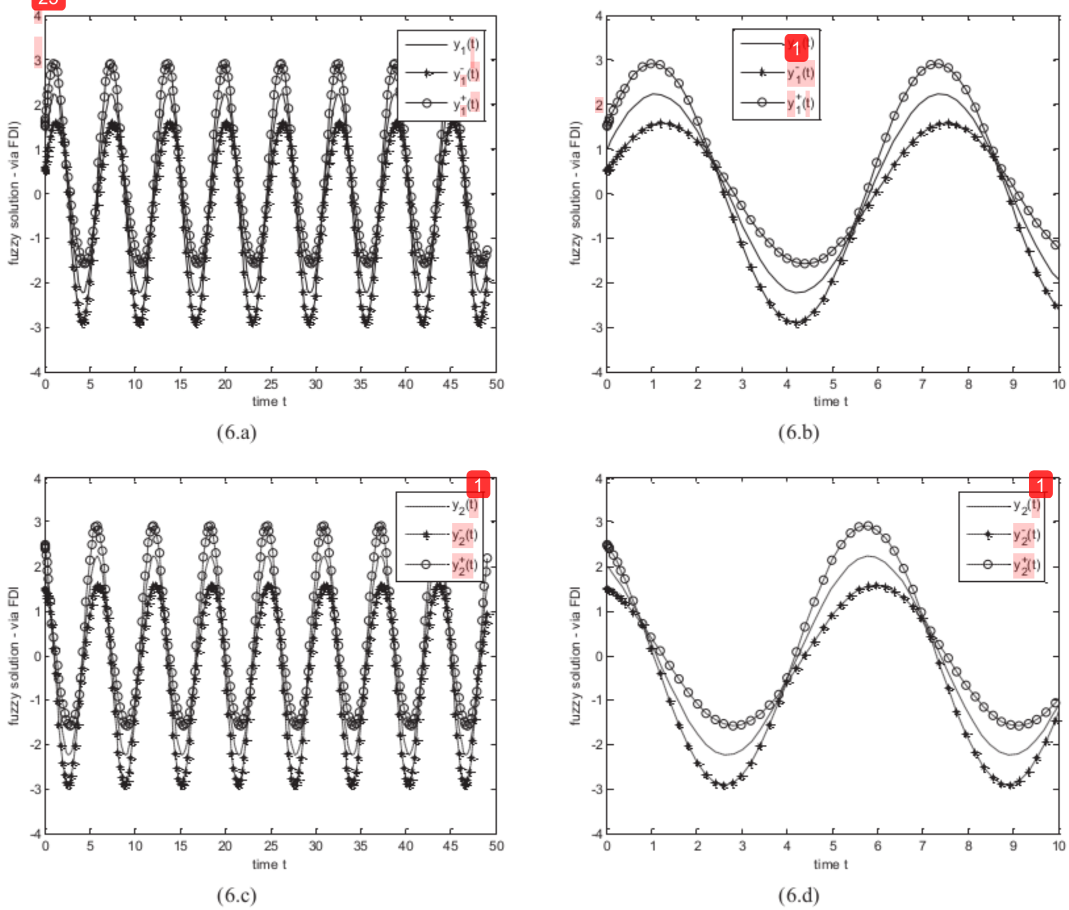


FIGURE 6. The α-cut of fuzzy solutions of the Problem (8) using FDI concept, with the parameters $k_1 = 1$, $k_2 = 0$. Information for (6.a), (6.b), (6.c) and (6.d) are equivalent to (5.a), (5.b), (5.c), and (5.d) in Fig 5, respectively.

CONCLUSION

In this work, three fuzzy solution concepts were examined to capture the oscillatory behavior of the harmonic oscillatior model.Two of them, namely HD and its generalization, were not able to capture the oscillations. In contrast, FDI concept was able to capture the oscillations and maintain the uncertainty of the solutions. Our present findings will give some ideas to study further the oscillatory behavior of a system containing uncertainty, that may often occur for biological systems.

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